

FOURIER COEFFICIENTS FOR DEGENERATE EISENSTEIN SERIES AND THE DESCENDING DECOMPOSITION

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ABSTRACT. We determine the unipotent orbits attached to degenerate Eisenstein series on general linear groups. This confirms a conjecture of David Ginzburg. This also shows that any unipotent orbit of general linear groups does occur as the unipotent orbit attached to a specific automorphic representation. The key ingredient is a root-theoretic result. To prove it, we introduce the notion of the descending decomposition, which expresses every Weyl group element as a product of simple reflections in a certain way. It is suitable for induction and allows us to translate the question into a combinatorial statement.

1. INTRODUCTION

Knowledge of Fourier coefficients is one of the most important tools in the theory of automorphic representations. Such Fourier coefficients are parameterized by unipotent orbits. In this article, we study Fourier coefficients for degenerate Eisenstein series on general linear groups.

Let F be a number field and \mathbb{A} be its adèle ring. Let G be a connected reductive group over F . Let \mathcal{O} denote a unipotent orbit of the group G . As explained in Ginzburg [6], one can associate to this unipotent orbit a unipotent subgroup $U_2(\mathcal{O})$ of G , and a family of characters of $U_2(\mathcal{O})(F) \backslash U_2(\mathcal{O})(\mathbb{A})$. Let $\psi_{U_2(\mathcal{O})}$ denote such a character. With this data, one can define a Fourier coefficient of an automorphic form f as the integral

$$\int_{U_2(\mathcal{O})(F) \backslash U_2(\mathcal{O})(\mathbb{A})} f(ug) \psi_{U_2(\mathcal{O})}(u) du. \quad (1)$$

The unipotent orbits are a partially ordered set, and for classical groups they are identified with partitions, based on the Jordan decomposition. Given an automorphic representation π on $G(\mathbb{A})$, we say that a unipotent orbit \mathcal{O} is attached to π if all Fourier coefficients of the form Eq. (1) for larger or incomparable orbits vanish identically and some coefficient for \mathcal{O} is nonzero. The determination of unipotent orbits attached to certain residue representations has played an important role in the descent method (see Ginzburg, Rallis and Soudry [10]). According to Ginzburg's dimension equation formalism on Rankin-Selberg integrals (Ginzburg [7, 8]), this information is also frequently useful in the construction of Eulerian global integrals. The unipotent orbits attached to an automorphic representation also have a connection to its Arthur parameter (see Jiang [12]).

In this paper we determine the unipotent orbits attached to degenerate Eisenstein series on general linear groups. From now on, let $G = \mathrm{GL}_r$. Let $\mu = (t_1 \cdots t_b)$ be a partition of r

Date: June 30, 2016.

2010 Mathematics Subject Classification. Primary 11F30, 11F70; Secondary 05E10, 22E50, 22E55.

Key words and phrases. Fourier coefficient, unipotent orbit, Eisenstein series, descending decomposition.

with $t_1 \geq \cdots \geq t_b > 0$. Let $\mu^\top = (r_1 \cdots r_a)$ denote the transpose of μ . Let $P = P_{\mu^\top}$ be the standard parabolic subgroup whose Levi subgroup is $M_{\mu^\top} \cong \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_a}$. Let δ_P be the modular quasicharacter of G with respect to P . We denote by $E_\mu(g, \underline{s})$ the Eisenstein series which corresponds to the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \delta_P^{\underline{s}}$. Here $\underline{s} = (s_1, \dots, s_a)$ denotes a multi-complex variable.

Theorem 1.1 (Theorem 5.12). *With the above notation, suppose that $\mathrm{Re}(s_i) \gg 0$. Then the unipotent orbit attached to $E_\mu(g, \underline{s})$ is μ .*

This confirms Conjecture 5.1 in Ginzburg [6]. A local analogue is given in Theorem 5.13. This also implies that any unipotent orbit of general linear groups can occur as the unipotent orbit attached to a specific automorphic representation.

The difficulty in the proof of Theorem 1.1 is to show the vanishing part for incomparable orbits, and the obstruction is of a combinatorial nature. When μ is of the form (a^b) , the situation is easier and a similar argument for symplectic groups can be found in Jiang and Liu [14] Lemma 3.1. Indeed, an orbit $(p_1 \cdots p_m)$ is greater than or not comparable to (a^b) if and only if $p_1 > a$. In other words, $(p_1 \cdots p_m)$ is greater than (a^b) in the lexicographical order. To generalize, we need to handle these incomparable orbits uniformly.

The key ingredient in our proof is a new root-theoretic result, which is connected to Theorem 1.1 via the study of semi-Whittaker coefficients. Let $\lambda = (p_1 \cdots p_k)$ be a partition of r (here we do not require $p_1 \geq \cdots \geq p_k$). Let P_λ be the standard parabolic subgroup of GL_r whose Levi subgroup $M_\lambda \cong \mathrm{GL}_{p_1} \times \cdots \times \mathrm{GL}_{p_k}$. Fix a nontrivial additive character $\psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$. Let $\psi_\lambda : U(F) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$ be the character such that it acts as ψ on the simple positive root subgroups contained in M_λ , and acts trivially otherwise. With this data, a λ -semi-Whittaker coefficient of an automorphic form f is

$$\int_{U(F) \backslash U(\mathbb{A})} f(ug) \psi_\lambda(u) du.$$

There is a strong relation between these two types of Fourier coefficients. In the local context, this is developed in Mœglin and Waldspurger [18], and Gomez, Gourevitch and Sahi [11]. We prove a global version in Proposition 5.3. Thus, to prove Theorem 1.1, it suffices to prove the following result.

Theorem 1.2 (Theorem 4.1). *Suppose $\mathrm{Re}(s_i) \gg 0$.*

(1) *If there is an index l such that $p_1 + \cdots + p_l > t_1 + \cdots + t_l$, then*

$$\int_{U(F) \backslash U(\mathbb{A})} E_\mu(ug, \underline{s}) \psi_\lambda(u) du = 0$$

for all choices of data.

(2) *The semi-Whittaker coefficient*

$$\int_{U(F) \backslash U(\mathbb{A})} E_\mu(ug, \underline{s}) \psi_\mu(u) du$$

is nonzero for some choice of data.

When $\operatorname{Re}(s_i) \gg 0$, by a standard unfolding argument, Theorem 1.2 is quickly reduced to the following root-theoretic result.

Theorem 1.3 (Theorem 3.1). *Let Δ denote the set of simple roots with respect to the standard Borel subgroup B . Let Φ^+ and Φ^- be the set of positive roots and negative roots, respectively. Let Δ_λ denote the set of simple roots contained in M_λ . Let $\Phi_{\mu^\top}^-$ denote the set of negative roots contained in M_{μ^\top} . Let $W(P_{\mu^\top})$ and $W(G)$ be the Weyl groups of P_{μ^\top} and G , respectively. Let $[W(P_{\mu^\top}) \backslash W(G)]$ be the set of minimal representatives for $W(P_{\mu^\top}) \backslash W(G)$.*

(1) *If there is an index l such that*

$$p_1 + \cdots + p_l > t_1 + \cdots + t_l,$$

then for any $w \in [W(P_{\mu^\top}) \backslash W(G)]$, there exists $\alpha \in \Delta_\lambda$ such that $w(\alpha) > 0$.

(2) *If $\lambda = \mu$, then there exists a unique $w_\mu \in [W(P_{\mu^\top}) \backslash W(G)]$ such that $w_\mu(\alpha) \in \Phi^- - \Phi_{\mu^\top}^-$ for all $\alpha \in \Delta_\mu$; and $w(\Delta_\mu) \cap \Phi^+ \neq \emptyset$ for all $w \neq w_\mu$.*

This result is analogous to Casselman and Shalika [4] Lemma 1.5. We prove it by translating the root-theoretic problem into a combinatorial problem. Thus we need to analyze the action of the Weyl group on the simple roots with care. To do so, we introduce the notion of descending decomposition, which is a systematic way to write down all the Weyl group elements. The descending decomposition expresses every element as a product of simple reflections in a certain way and it is suitable for induction. To simplify notations we let $G = \operatorname{GL}_{r+1}$ and $W(G)$ be its Weyl group. Choose the following long word decomposition

$$w_0 = s_1(s_2s_1)(s_3s_2s_1) \cdots (s_rs_{r-1} \cdots s_1),$$

where s_1, \dots, s_r are simple reflections in $W(G)$ with the usual labeling. In fact, the expression $s_i s_{i-1} \cdots s_1$ is the long word in the set of minimal representatives for $W(\operatorname{GL}_i) \backslash W(\operatorname{GL}_{i+1})$. Choosing a string starting with s_i in each $(s_i s_{i-1} \cdots s_1)$, and multiplying them, we obtain an element in $W(G)$. Surprisingly, every $w \in W(G)$ has a unique expression of this form.

To be more precise, let

$$D = \{(k_1, \dots, k_r) : k_i \in \mathbb{Z}, 1 \leq k_i \leq i+1\}.$$

For each k_i , define

$$\pi_{k_i} = \begin{cases} s_i s_{i-1} \cdots s_{k_i} & \text{if } 1 \leq k_i \leq i, \\ e \text{ (the identity element)} & \text{if } k_i = i+1. \end{cases} \quad (2)$$

Define a map

$$\pi : D \rightarrow W(G), \quad (k_1, \dots, k_r) \mapsto \pi_{k_1} \cdots \pi_{k_r}.$$

Proposition 1.4. (Proposition 2.2) *The map π is a bijection. Moreover, $\pi(k_1, \dots, k_r)$ is reduced.*

This leads to a natural way to express elements in $[W(P_{\mu^\top}) \backslash W(G)]$ as products of simple reflections (Theorem 2.4). It allows us to compute the action of the Weyl group on the set of simple roots systematically, and translate Theorem 1.3 into a combinatorial fact.

Finally we remark that for other Cartan types, the situation is more complicated. For classical groups, it is known that only special unipotent orbits can occur as the unipotent

orbits attached to automorphic representations (see [6] Theorem 3.1, [15]) and further assumptions are required for cuspidal representations (see [6] Section 4, [9] Theorem 2.7). The interested reader is also referred to [6] Section 5.2 for further information.

The rest of this paper is organized as follows. Section 2 introduces the descending decomposition and constructs a set of coset representatives. Section 3 is devoted to proving the root-theoretic result. Section 4 introduces degenerate Eisenstein series and proves Theorem 4.1. We introduce Fourier coefficients associated with unipotent orbits in Section 5.1, and establish the connection between these two types of Fourier coefficients in Section 5.2. The main result, i.e. the determination of the unipotent orbits attached to degenerate Eisenstein series, is given in Theorem 5.12.

Acknowledgement. The descending decomposition was first discovered by Mark Reeder in a different context. The author would like to thank him for many helpful discussions. The author would also like to thank Solomon Friedberg and David Ginzburg for the encouragement and their helpful comments. This work was supported by the National Science Foundation, grant number 1500977.

2. DESCENDING DECOMPOSITION AND COSET REPRESENTATIVES

Let $G = \mathrm{GL}_{r+1}$. Let B denote the standard Borel subgroup of G with torus T and unipotent radical U . Let Φ be the root system of G . Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ denote the set of simple roots with respect to B (with the usual labeling). Let Φ^+ and Φ^- be the set of positive roots and negative roots, respectively. For each simple root $\alpha_i \in \Delta$, let s_i denote the corresponding simple reflection. The Weyl group $W(G)$ of G is generated by s_1, \dots, s_r and $W(G) \cong S_{r+1}$.

2.1. Descending Decomposition. Choose the following *nice* long word decomposition (in the sense of Littelmann [16] Section 4)

$$w_0 = s_1(s_2s_1)(s_3s_2s_1) \cdots (s_rs_{r-1} \cdots s_1).$$

Let

$$D = \{(k_1, \dots, k_r) : k_i \in \mathbb{Z}, 1 \leq k_i \leq i+1\}.$$

For each k_i , define

$$\pi_{k_i} = \begin{cases} s_i s_{i-1} \cdots s_{k_i} & \text{if } 1 \leq k_i \leq i, \\ e \text{ (the identity element)} & \text{if } k_i = i+1. \end{cases} \quad (3)$$

Each π_{k_i} is called a *cycle* (notice this is slightly stronger than the usual “cycle” in S_{r+1}). For convenience, sometimes we use $s_{i_1 i_2 \dots i_N}$ to denote $s_{i_1} s_{i_2} \cdots s_{i_N}$ in each cycle. Define a map

$$\pi : D \rightarrow W(G), \quad (k_1, \dots, k_r) \mapsto \pi_{k_1} \cdots \pi_{k_r}.$$

Example 2.1. If $r = 2$, then the Weyl group is S_3 . The long word decomposition we consider is

$$w_0 = (s_1)(s_2s_1).$$

It is easy to check:

$$\begin{aligned} \pi(2, 3) &= e, & \pi(2, 2) &= s_2, & \pi(2, 1) &= s_{21}, \\ \pi(1, 3) &= s_1, & \pi(1, 2) &= s_1 s_2, & \pi(1, 1) &= s_1 s_{21}. \end{aligned}$$

These are all the elements in S_3 .

Proposition 2.2. *The map $\pi : D \rightarrow W(G)$ is a bijection. Moreover, $\pi(k_1, \dots, k_r)$ is reduced.*

Proof. We first prove that the map π is surjective. The case $r = 1$ is clear. We assume the result is true for $r - 1$, and prove it for r . Any Weyl group element w can be written as a product of simple reflections. Choose a reduced expression for w such that the number of s_r is a minimum. If the expression does not contain s_r , then we are done by induction. If it contains s_r , we show that it contains at most one s_r . Suppose there are two. By applying induction on the expression between the two s_r 's, w can be written in one of the following forms:

$$\cdots s_r \cdots s_{r-1} \cdots s_r \cdots, \text{ or } \cdots s_r \cdots s_r \cdots$$

Notice that s_r commutes with all the simple reflections except s_{r-1} . If there is no s_{r-1} , then the two s_r 's cancel with each other; if there is only one s_{r-1} , then we can use the relation $s_r s_{r-1} s_r = s_{r-1} s_r s_{r-1}$ to reduce the number of s_r . Thus there is at most one s_r in the reduced expression.

Now applying induction to the expression on the right-hand side of s_r , we obtain a product of cycles. Then in the reduced expression we can move every cycle on the right-hand side of s_r to the left-hand side except the last cycle ($s_{r-1} s_{r-2} \cdots$). Then apply induction again on the left-hand side to obtain the desired expression. This shows that the map π is surjective.

Notice that the cardinalities of D and $W(G)$ are both $(r + 1)!$. Thus π is a bijection.

Now we show that $\pi(k_1, \dots, k_r)$ is a reduced expression. Let $\Phi^-(w) = \{\alpha > 0 : w\alpha < 0\}$. If α is a simple root, then

$$\ell(ws_\alpha) = \begin{cases} \ell(w) + 1, & \text{if } \alpha \notin \Phi^-(w), \\ \ell(w) - 1, & \text{if } \alpha \in \Phi^-(w). \end{cases}$$

Thus, it suffices to show that

$$\pi_{k_1} \cdots \pi_{k_{t-1}}(\alpha_t + \cdots + \alpha_j) > 0$$

for any k_1, \dots, k_{t-1} and all j , $1 \leq j \leq t$.

Notice that $\pi_{k_1} \cdots \pi_{k_{t-1}} \in \langle s_1, \dots, s_{t-1} \rangle$. The coefficient of α_t in $\pi_{k_1} \cdots \pi_{k_{t-1}}(\alpha_t + \cdots + \alpha_j)$ must be 1. This shows that $\pi_{k_1} \cdots \pi_{k_{t-1}}(\alpha_t + \cdots + \alpha_j)$ is a positive root. \square

Remark 2.3. If we identify $W(G)$ with the permutation group on $r + 1$ elements such that s_i corresponds to the transposition $(i, i + 1)$, then

$$\pi_{k_i} = s_i s_{i-1} \cdots s_{k_i} \longleftrightarrow (i + 1, i, \dots, k_i) \in S_{r+1}.$$

This explains the title *descending decomposition*; see also Reeder [19] Section 3.2.1 for an explicit construction of the inverse map and Section 7.1 for the natural appearance of the descending decomposition in the Bruhat decomposition of GL_{r+1} for the upper triangular Borel subgroup.

2.2. Coset Representatives. Let $(r_1 \cdots r_a)$ be a general partition of $r + 1$, i.e. we do not require $r_1 \geq \cdots \geq r_a$. Let P denote the standard parabolic subgroup of G whose Levi subgroup is $\text{GL}_{r_1} \times \cdots \times \text{GL}_{r_a}$ embedded in G via

$$\text{GL}_{r_1} \times \cdots \times \text{GL}_{r_a} \rightarrow G : (g_1, \dots, g_a) \mapsto \text{diag}(g_1, \dots, g_a).$$

We use *Young tableaux* to describe a nice set of coset representatives for $W(P) \backslash W(G)$. For the partition $(r_1 \cdots r_a)$, we require its Young diagram to have r_i boxes in the i th column.

Notice that this is not the usual definition and we do *not* require $r_1 \geq \cdots \geq r_a$. (The definition in Section 3 is different but they are consistent.) We fill in the boxes in the Young diagram with k_0, k_1, \dots, k_r from top to bottom in columns, from the leftmost column to the rightmost column. For example, if the partition is (342), then the Young tableau is

k_0	k_3	k_7
k_1	k_4	k_8
k_2	k_5	
	k_6	

Notice that if we delete the first row, then the simple reflections with the remaining indices generate $W(P)$.

Now delete the entries in the first column. Define $D_{(r_1 \dots r_a)}$ to be the set of such Young tableaux with the following two conditions:

- (a) $k_i \in \mathbb{Z}$ and $1 \leq k_i \leq i + 1$.
 - (b) The entries in each column are strictly increasing.
- (4)

For each i , define π_{k_i} by Eq. (3). Then condition (b) means that the lengths of π_{k_i} in each column are non-increasing. For convenience, we usually write Π_j for the product of π_{k_i} 's in the j th column, although this depends on the choices of k_i 's.

Given a Young tableau $Y \in D_{(r_1 \dots r_a)}$, by taking the product of π_{k_i} in all the columns, we can define a Weyl group element $\Pi_2 \cdots \Pi_a \in W(G)$. This defines a map $\pi_{(r_1 \dots r_a)} : D_{(r_1 \dots r_a)} \rightarrow W(P) \setminus W(G)$.

Theorem 2.4. *The map $\pi_{(r_1 \dots r_a)}$ is a bijection.*

Example 2.5. If the partition is (32), then the Young tableau is

k_0	k_3
k_1	k_4
k_2	

Deleting the entries in first column gives

	k_3
	k_4

The assumptions are

$$1 \leq k_3 \leq 4, \quad 1 \leq k_4 \leq 5, \quad k_3 < k_4.$$

All the elements of the form $\pi_{k_3}\pi_{k_4}$ give a set of coset representatives for the quotient $W(\mathrm{GL}_3 \times \mathrm{GL}_2) \backslash W(\mathrm{GL}_5)$. There are 10 elements:

$$\begin{aligned} &e \\ &s_3, \quad s_3s_4, \\ &s_{32}, \quad s_{32}s_4, \quad s_{32}s_{43}, \\ &s_{321}, s_{321}s_4, s_{321}s_{43}, \quad s_{321}s_{432}. \end{aligned}$$

Example 2.6. If the partition is (323), then the Young tableau is

k_0	k_3	k_5
k_1	k_4	k_6
k_2		k_7

Deleting the entries in the first column gives

	k_3	k_5
	k_4	k_6
		k_7

Then the assumptions are

$$\begin{aligned} 1 \leq k_3 \leq 4, \quad 1 \leq k_4 \leq 5, \quad k_3 < k_4, \\ 1 \leq k_5 \leq 6, \quad 1 \leq k_6 \leq 7, \quad 1 \leq k_7 \leq 8, \quad k_5 < k_6 < k_7. \end{aligned}$$

All the elements of the form $(\pi_{k_3}\pi_{k_4})(\pi_{k_5}\pi_{k_6}\pi_{k_7})$ where the lengths of π_{k_3} and π_{k_4} (resp. π_{k_5} , π_{k_6} and π_{k_7}) are non-increasing give a set of coset representatives for $W(P) \backslash W(G)$.

We need the following lemma for the proof of Theorem 2.4.

Lemma 2.7. *If $i \geq j$, then*

$$(s_i \cdots s_j)(s_{i+1} \cdots s_j) = s_{i+1}(s_i \cdots s_j)(s_{i+1} \cdots s_{j+1}).$$

Moreover, if $i \geq j \geq k$, then

$$(s_i \cdots s_j)(s_{i+1} \cdots s_k) = s_{i+1}(s_i \cdots s_k)(s_{i+1} \cdots s_{j+1}).$$

Proof. The second statement follows immediately from the first statement. We prove the first statement by induction on $i - j$. When $i - j = 0$, the result is clear since $(s_i)(s_{i+1}s_i) = s_{i+1}(s_i)(s_{i+1})$. For general i, j , notice that

$$\begin{aligned} &(s_i \cdots s_j)(s_{i+1} \cdots s_j) \\ &= (s_i \cdots s_{j+1})s_j(s_{i+1} \cdots s_{j+2})s_{j+1}s_j \\ &= (s_i \cdots s_{j+1})(s_{i+1} \cdots s_{j+2})s_js_{j+1}s_j \\ &= (s_i \cdots s_{j+1})(s_{i+1} \cdots s_{j+2})s_{j+1}s_js_{j+1} \\ &= s_{i+1}(s_i \cdots s_{j+1})(s_{i+1} \cdots s_{j+2})s_js_{j+1} \text{ (by induction)} \\ &= s_{i+1}(s_i \cdots s_{j+1}s_j)(s_{i+1} \cdots s_{j+2}s_{j+1}). \end{aligned}$$

This finishes the proof. □

Proof of Theorem 2.4. Let $w \in W(P) \setminus W(G)$. We need to choose a coset representative satisfying conditions (a) and (b) in Eq. (4). By Proposition 2.2, we can choose a coset representative satisfying condition (a). We only need to show (b). If π_{k_i} and $\pi_{k_{i+1}}$ are in the same row of the Young tableau, and $k_i \geq k_{i+1}$, then by Lemma 2.7,

$$\pi_{k_i} \pi_{k_{i+1}} = s_{i+1} \pi_{k'_i} \pi_{k'_{i+1}}$$

where $k'_i = k_{i+1}$, $k'_{i+1} = k_i + 1$. Notice that $s_{i+1} \in W(P)$ and $k'_{i+1} > k'_i$. Therefore, we can always choose a coset representative satisfying both conditions.

To show that map is bijective, we again count the cardinalities of both sides. This is a combinatorial exercise and the proof is left to the reader. \square

By Casselman [3] Lemma 1.1.2, there is a unique element of minimal length in each coset of $W(P) \setminus W(G)$. The coset representative $\Pi_2 \cdots \Pi_a$ is indeed this unique element in its coset. For completeness, we give a proof here.

Proposition 2.8. *The element $\Pi_2 \cdots \Pi_a$ is of minimal length in its coset in $W(P) \setminus W(G)$.*

Proof. We show that the coset representatives in Theorem 2.4 are of minimal lengths in each coset. We only have to show that for each w , $w^{-1}(\Delta_\lambda) > 0$ ([3] Section 1). Equivalently, we need to show that, if $\alpha_l \in \Delta_\lambda$ (in other words, $l \neq r_1, r_1 + r_2, \dots, r_1 + \dots + r_{k-1}$), then $\alpha_l \notin \Phi^-(w^{-1})$.

Recall that if $w = s_{i_1} \cdots s_{i_N}$ is a reduced decomposition, then

$$\{\alpha > 0 : w\alpha < 0\} = \{\alpha_{i_N}, s_{i_N} \alpha_{i_{N-1}}, \dots, s_{i_N} s_{i_{N-1}} \cdots s_{i_2} \alpha_{i_1}\}.$$

Write $w = \pi_{k_{r_1}} \cdots \pi_{k_r} \in W(P) \setminus W(G)$. Thus it suffices to show that, given $\alpha_l \in \Delta_\lambda$,

$$\pi_{k_{r_1}} \cdots \pi_{k_{j+1}}(\alpha_j + \dots + \alpha_k) \neq \alpha_l \quad (5)$$

for all $r_1 \leq j \leq r$ and all possible k . Indeed, if $j < l$, then α_l cannot appear on the left-hand side. If $j > l$, then the coefficient of α_j in the left-hand side is 1. In these two cases, Eq. (5) is true. If $j = l$, then $k_{l-1} < k_l$ (since $\alpha_l \in \Delta_\lambda$ and k_l is not on the top row in the Young tableau). Therefore, for $k \geq k_l$,

$$(s_{l-1} \cdots s_{k_{l-1}})(\alpha_l + \dots + \alpha_k) = \alpha_l + \alpha_{l-1} + \dots + \alpha_k + \alpha_{k-1}.$$

The expression before $(s_{l-1} \cdots s_{k_{l-1}})$ is in $\langle s_1, \dots, s_{l-2} \rangle$. Thus the coefficients of α_{l-1}, α_l in $\pi_{k_{r_1}} \cdots \pi_{k_{j+1}}(\alpha_j + \dots + \alpha_k)$ are both 1. Thus Eq. (5) also holds. This completes the proof. \square

3. ROOT-THEORETIC RESULTS

The goal of this section is to prove a root-theoretic result, which is used in the proof of Theorem 4.1.

Fix a partition $\mu = (t_1 \cdots t_b)$ of $r+1$ such that $t_1 \geq \dots \geq t_b > 0$. Let $\mu^\top = (r_1 \cdots r_a)$ be its transpose. Let P_{μ^\top} be the parabolic subgroup whose Levi subgroup $M_{\mu^\top} \cong \mathrm{GL}_{r_1} \times \dots \times \mathrm{GL}_{r_a}$. We represent $w \in W(P_{\mu^\top}) \setminus W(G)$ by using the coset representatives in Theorem 2.4.

Let $\lambda = (p_1 \cdots p_m)$ be a general partition of $r+1$. Let $\lambda^\top = (q_1 \cdots q_n)$ be its transpose. Here $q_i = \#\{j : p_j \geq i\}$. In particular, $q_1 = m$. Let P_λ denote the standard parabolic subgroup of G whose Levi subgroup is $M_\lambda \cong \mathrm{GL}_{p_1} \times \dots \times \mathrm{GL}_{p_m}$.

Now we are ready to state the main result in this section.

Theorem 3.1. Let Δ_λ denote the set of simple roots contained in M_λ . Let $\Phi_{\mu^\top}^-$ denote the set of negative roots contained in M_{μ^\top} .

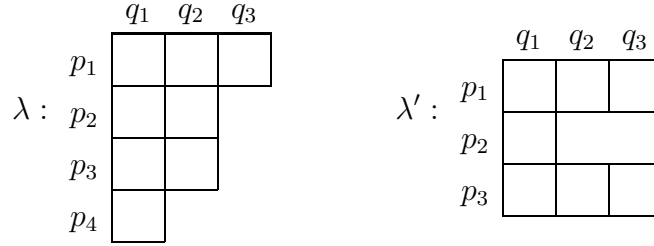
(1) If there is an index l such that

$$p_1 + \cdots + p_l > t_1 + \cdots + t_l,$$

then for any $w \in W(P_{\mu^\top}) \setminus W(G)$, there exists $\alpha \in \Delta_\lambda$ such that $w(\alpha) > 0$.

(2) If $\lambda = \mu$, then there exists a unique $w_\mu \in W(P_{\mu^\top}) \setminus W(G)$ such that $w_\mu(\alpha) \in \Phi^- - \Phi_{\mu^\top}^-$ for all $\alpha \in \Delta_\mu$; and $w(\Delta_\mu) \cap \Phi^+ \neq \emptyset$ for all $w \neq w_\mu$.

3.1. Transpose of partitions. The relation between λ and λ^\top can be easily read from the Young diagram associated with λ . From now on, the Young diagram associated with λ is the one that has m rows, and p_i boxes in the i th row. Here the definition also applies to general partitions, i.e. we do not assume the row sizes are weakly decreasing. We also label the rows and columns with the entries they represent. Here are examples with $\lambda = (3221)$, $\lambda^\top = (431)$ and $\lambda' = (313)$, $\lambda'^\top = (322)$:



Lemma 3.2.

(1) For $1 \leq l \leq m$ and $1 \leq k \leq n$,

$$p_1 + \cdots + p_l \leq l \cdot k + q_{k+1} + \cdots + q_n.$$

(2) If furthermore, $p_1 \geq \cdots \geq p_m > 0$ and $k = p_{l+1}$, then equality holds in (1).

Proof. (1) For $1 \leq l \leq m$ and $1 \leq k \leq n$,

$$\begin{aligned}
& p_1 + \cdots + p_l \\
& \leq \sum_{i=1}^n \min(q_i, l) \\
& = \sum_{i=1}^k \min(q_i, l) + \sum_{i=k+1}^n \min(q_i, l) \\
& \leq l \cdot k + q_{k+1} + \cdots + q_n.
\end{aligned}$$

(2) If $p_1 \geq \cdots \geq p_m > 0$, then $q_i = \max\{j : p_j \geq i\}$ and

$$p_1 + \cdots + p_l = \sum_{i=1}^n \min(q_i, l).$$

If $k = p_{l+1} = \max\{j : q_j \geq l+1\}$, then

$$\min(q_i, l) = \begin{cases} l & \text{if } 1 \leq i \leq p_{l+1}, \\ q_i & \text{if } i \geq p_{l+1} + 1. \end{cases}$$

Thus the equality holds. \square

3.2. The action on simple roots. Let us start with the following observations for general $w \in W(G)$. We represent w by using the descending decomposition in Proposition 2.2.

Lemma 3.3. *If $i \leq j$, then*

$$s_j s_{j-1} \cdots s_i(\alpha_k) = \begin{cases} \alpha_k & \text{if } k \leq i-2 \text{ or } k \geq j+2, \\ \alpha_{i-1} + \alpha_i + \cdots + \alpha_j & \text{if } k = i-1, \\ -(\alpha_i + \cdots + \alpha_j) & \text{if } k = i, \\ \alpha_{k-1} & \text{if } j \geq k \geq i+1, \\ \alpha_j + \alpha_{j+1} & \text{if } k = j+1. \end{cases}$$

Proof. A straightforward calculation. \square

Lemma 3.4. *Suppose $i \leq j$. If $\pi_{k_j}(\alpha_l)$ is a negative root, so is $\pi_{k_i} \cdots \pi_{k_j}(\alpha_l)$.*

Proof. If $\pi_{k_j}(\alpha_l)$ is negative, then the coefficient of α_j in $\pi_{k_j}(\alpha_l)$ is -1 . Notice that the element $\pi_{k_i} \cdots \pi_{k_{j-1}} \in \langle s_1, \dots, s_{j-1} \rangle$. Thus the coefficient of α_j in $\pi_{k_i} \cdots \pi_{k_j}(\alpha_l)$ is also -1 . This implies that $\pi_{k_i} \cdots \pi_{k_j}(\alpha_l) < 0$. \square

Lemma 3.5. *Suppose $i \leq j$. If $\pi_{k_j}(\alpha_l)$ is a positive root but not a simple root, then $\pi_{k_i} \cdots \pi_{k_j}(\alpha_l) > 0$.*

Proof. This follows by the same argument as in Lemma 3.4. \square

Lemma 3.6. *If $i \leq j$, then*

$$\#\{\alpha \in \Delta : \pi_{k_i} \cdots \pi_{k_j}(\alpha) < 0\} \leq j - i + 1.$$

Proof. We prove this by induction on the number of cycles. When $j - i + 1 = 1$, the lemma follows from Lemma 3.3.

Notice that

$$\pi_{k_j}(\Delta) \subset \Delta \sqcup (\pi_{k_j}(\Delta) \cap \Phi^-) \sqcup (\pi_{k_j}(\Delta) \cap \Phi^+ \setminus \Delta).$$

By Lemmas 3.3 and 3.4, $\pi_{k_i} \cdots \pi_{k_{j-1}}(\pi_{k_j}(\Delta) \cap \Phi^-)$ consists of at most one negative root, and any element in $\pi_{k_i} \cdots \pi_{k_{j-1}}(\pi_{k_j}(\Delta) \cap \Phi^+ \setminus \Delta)$ is always positive.

Thus

$$\begin{aligned} & \pi_{k_i} \cdots \pi_{k_j}(\Delta) \cap \Phi^- \\ &= \pi_{k_i} \cdots \pi_{k_{j-1}}(\pi_{k_j}(\Delta) \cap \Phi^-) \\ &\subset (\pi_{k_i} \cdots \pi_{k_{j-1}}(\Delta) \cap \Phi^-) \sqcup \pi_{k_i} \cdots \pi_{k_{j-1}}(\pi_{k_j}(\Delta) \cap \Phi^-). \end{aligned}$$

This finishes the induction step. \square

Lemma 3.7. *Suppose $i \leq j$. Consider $w = \pi_{k_i} \cdots \pi_{k_j}$ such that the sequence k_i, \dots, k_j is strictly increasing. For $k_j < l \leq j$, $w(\alpha_l) = \alpha_{l+i-j-1}$.*

Proof. Notice that

$$k_{j-1} < l-1, k_{j-2} < l-2, \dots, k_i < l-j+i.$$

Thus,

$$w(\alpha_l) = \pi_{k_i} \cdots \pi_{k_j}(\alpha_l) = \pi_{k_i} \cdots \pi_{k_{j-1}}(\alpha_{l-1}) = \cdots = \alpha_{l+i-j-1}.$$

\square

Lemma 3.8. Consider $w = \Pi_2 \cdots \Pi_a \in W(P) \setminus W(G)$ and write $\Pi_a = \pi_{k_i} \cdots \pi_{k_j}$. Let $\alpha_m, \alpha_{m+1}, \dots, \alpha_n$ be a sequence of $n - m + 1$ simple roots. Then at least one of the following statements is true.

- (1) At least one of $w(\alpha_m), \dots, w(\alpha_n)$ is positive.
- (2) $\Pi_a(\alpha_m), \dots, \Pi_a(\alpha_n)$ is still a sequence of consecutive simple roots.
- (3) $w(\alpha_m) < 0$ and $\Pi_a(\alpha_{m+1}), \dots, \Pi_a(\alpha_n)$ is again a sequence of $n - m$ consecutive simple roots.

Proof. We prove this by induction on the number of cycles in Π_a . If $j - i + 1 = 1$, then one of the following holds:

- (a) If $m + 1 \leq k_j \leq n + 1$, then $\pi_{k_j}(\alpha_{k_j-1}) > 0$ but it is not a simple root. This implies $w(\alpha_{k_j-1}) > 0$.
- (b) If $m - 1 \leq j \leq n - 1$, then $\pi_{k_j}(\alpha_{j+1}) > 0$ but it is not a simple root. And similarly $w(\alpha_{j+1}) > 0$.
- (c) If $k_j \geq n + 2$, or $m \geq j + 2$, then $(\Pi_a(\alpha_m), \dots, \Pi_a(\alpha_n)) = (\alpha_m, \dots, \alpha_n)$.
- (d) If $k_j = m$ and $j \geq n$, then $\Pi_a(\alpha_m) < 0$, and $(\Pi_a(\alpha_{m+1}), \dots, \Pi_a(\alpha_n)) = (\alpha_m, \dots, \alpha_{n-1})$.
- (e) If $k_j + 1 \leq m \leq n \leq j$, then $(\Pi_a(\alpha_m), \dots, \Pi_a(\alpha_n)) = (\alpha_{m-1}, \dots, \alpha_{n-1})$.

Now we analyze the induction process. Write $\Pi_a = \pi_{k_i} \cdots \pi_{k_{j-1}} \cdot \pi_{k_j}$ and compute the action of π_{k_j} first. There are 3 cases.

- (i) In cases (c) and (e), we are done by induction.
- (ii) In cases (a) and (b), we are done by Lemma 3.5.
- (iii) If we are in case (d), then by Lemma 3.4, we have $w(\alpha_m) < 0$. By Lemma 3.7, $\Pi_a(\alpha_{m+l}) = \alpha_{m+l+i-j-1}$. Therefore, $\Pi_a(\alpha_{m+1}), \dots, \Pi_a(\alpha_n)$ is still a sequence of $n - m$ consecutive simple roots.

□

This immediately implies the following.

Lemma 3.9. Let $w = \Pi_2 \cdots \Pi_a \in W(P) \setminus W(G)$. Let $\alpha_m, \alpha_{m+1}, \dots, \alpha_n$ be a sequence of $n - m + 1$ simple roots. If $n - m + 1 > a - 1$, then there is some $m \leq l \leq n$ such that $w(\alpha_l) > 0$.

3.3. Proof of Theorem 3.1 part (1). Recall we have a partition $\lambda = (p_1 \cdots p_m)$. We fill in the Young diagram for λ with $\alpha_1, \dots, \alpha_r$ from right to left in rows, from the top row to the bottom row. Deleting the first column gives us Δ_λ . For example, if $\lambda = (442)$ and $\lambda^\top = (3322)$, then we have

	q_1	q_2	q_3	q_4
p_1		α_3	α_2	α_1
p_2		α_7	α_6	α_5
p_3		α_9		

If $\lambda = (413)$ and $\lambda^\top = (3221)$, then we have

	q_1	q_2	q_3	q_4
p_1		α_3	α_2	α_1
p_2				
p_3		α_7	α_6	

For $1 \leq i \leq m$, let $\Delta_{\lambda,i}$ denote the set of simple roots contained in the i th row. It is a sequence of simple roots with consecutive indices and

$$\Delta_\lambda = \bigsqcup_{1 \leq i \leq m} \Delta_{\lambda,i}.$$

For $2 \leq i \leq n$, we also define Q_i to be the set of simple roots contained in the i th column. Thus,

$$\Delta_\lambda = \bigsqcup_{2 \leq i \leq n} Q_i.$$

From now on suppose there exists $w \in W(P_{\mu^\top}) \setminus W(G)$ such that $w(\alpha) < 0$ for all $\alpha \in \Delta_\lambda$. We shall derive a contradiction. Write $w = \Pi_2 \cdots \Pi_a$. Then for any $2 \leq l \leq a$ and $\alpha \in \Delta_\lambda$, $\Pi_l \cdots \Pi_a(\alpha)$ is either negative or a simple root (Lemma 3.5). For each simple root $\alpha \in \Delta_\lambda$, there is a smallest index l such that $\Pi_{l+1} \cdots \Pi_a(\alpha)$ is a simple root but $\Pi_l \cdots \Pi_a(\alpha) < 0$. We write R_l for the set of such simple roots (notice that this depends on w). Thus

$$\Delta_\lambda = \bigsqcup_{2 \leq l \leq a} R_l.$$

From Lemma 3.8 we deduce the following result.

Lemma 3.10. *With the above assumption, if $\alpha_i \in R_l$ and $\alpha_{i+1} \in \Delta_\lambda$, then $\alpha_{i+1} \in R_2 \cup \cdots \cup R_{l-1}$; if $\alpha_i \in R_2$, then $\alpha_{i+1} \notin \Delta_\lambda$.*

Before we prove the most general result, we give three examples to illustrate the ideas of our proof.

Example 3.11. Let $\mu = (3^2)$, $\mu^\top = (2^3)$ and $\lambda = (411)$. The corresponding Young tableaux are

					q_1	q_2	q_3	q_4	
$\mu :$		r_1	r_2	r_3	p_1		α_3	α_2	α_1
	t_1		k_2	k_4	p_2				
	t_2		k_3	k_5	p_3				

In this case, $G = \text{GL}_6$ and $P_{\mu^\top} \cong \text{GL}_2 \times \text{GL}_2 \times \text{GL}_2$. Then $W(P_{\mu^\top}) \setminus W(G)$ consists of elements of the form

$$(\pi_{k_2} \pi_{k_3})(\pi_{k_4} \pi_{k_5}),$$

where the cycles (π_{k_2}, π_{k_3}) and (π_{k_4}, π_{k_5}) are both descending. The result is clear from Lemma 3.9.

Example 3.12. Let $\mu = (4221)$, $\mu^\top = (4311)$ and $\lambda = (3^3)$. Notice that

$$p_1 + p_2 + p_3 = 3 + 3 + 3 > 4 + 2 + 2 = t_1 + t_2 + t_3.$$

The associated Young tableaux are

$$\mu : \begin{array}{c} r_1 \quad r_2 \quad r_3 \quad r_4 \\ t_1 \begin{array}{|c|c|c|c|} \hline & k_4 & k_7 & k_8 \\ \hline \end{array} \\ t_2 \begin{array}{|c|c|} \hline & k_5 \\ \hline \end{array} \\ t_3 \begin{array}{|c|c|} \hline & k_6 \\ \hline \end{array} \\ t_4 \begin{array}{|c|} \hline \\ \hline \end{array} \end{array} \quad \lambda : \begin{array}{c} q_1 \quad q_2 \quad q_3 \\ p_1 \begin{array}{|c|c|c|} \hline & \alpha_2 & \alpha_1 \\ \hline \end{array} \\ p_2 \begin{array}{|c|c|c|} \hline & \alpha_5 & \alpha_4 \\ \hline \end{array} \\ p_3 \begin{array}{|c|c|c|} \hline & \alpha_8 & \alpha_7 \\ \hline \end{array} \end{array}.$$

Notice $r_1 = 4$. Thus for any $w \in W(P_{\mu^\top}) \setminus W(G)$, there are at most $9 - 4 = 5$ cycles. However, $q_1 = 3$ and Δ_λ has $9 - 3 = 6$ simple roots. By Lemma 3.6, $w(\Delta_\lambda) \cap \Phi^+ \neq \emptyset$.

Example 3.13. Let $\mu = (42222)$, $\mu^\top = (5511)$ and $\lambda = (333111)$. Notice that

$$p_1 + p_2 + p_3 = 3 + 3 + 3 > 4 + 2 + 2 = t_1 + t_2 + t_2, \quad (6)$$

but $r_1 = 5 < 6 = q_1$. We cannot apply the argument in Example 3.12. The associated Young tableaux are

$$\mu : \begin{array}{c} r_1 \quad r_2 \quad r_3 \quad r_4 \\ t_1 \begin{array}{|c|c|c|c|} \hline & k_5 & k_{10} & k_{11} \\ \hline \end{array} \\ t_2 \begin{array}{|c|c|} \hline & k_6 \\ \hline \end{array} \\ t_3 \begin{array}{|c|c|} \hline & k_7 \\ \hline \end{array} \\ t_4 \begin{array}{|c|c|} \hline & k_8 \\ \hline \end{array} \\ t_5 \begin{array}{|c|c|} \hline & k_9 \\ \hline \end{array} \end{array} \quad \lambda : \begin{array}{c} q_1 \quad q_2 \quad q_3 \\ p_1 \begin{array}{|c|c|c|} \hline & \alpha_2 & \alpha_1 \\ \hline \end{array} \\ p_2 \begin{array}{|c|c|c|} \hline & \alpha_5 & \alpha_4 \\ \hline \end{array} \\ p_3 \begin{array}{|c|c|c|} \hline & \alpha_8 & \alpha_7 \\ \hline \end{array} \\ p_4 \begin{array}{|c|} \hline \\ \hline \end{array} \\ p_5 \begin{array}{|c|} \hline \\ \hline \end{array} \\ p_6 \begin{array}{|c|} \hline \\ \hline \end{array} \end{array}.$$

Recall that

$$R_2 = \{\alpha \in \Delta_\lambda : \Pi_3 \Pi_4(\alpha) \in \Delta, \Pi_2 \Pi_3 \Pi_4(\alpha) < 0\}.$$

We claim that $R_2 \subset Q_2 = \{\alpha_2, \alpha_5, \alpha_8\}$. If not, then R_2 contains at least one of $\alpha_1, \alpha_4, \alpha_7$. Without loss of generality, we assume $\alpha_1 \in R_2$. By Lemma 3.10, we deduce that $\alpha_2 \notin \Delta_\lambda$. This is a contradiction.

We now conclude that $R_2 \subset Q_2$ and $R_3 \sqcup R_4 \supset Q_3$. By Lemma 3.6, the number of cycles in $\Pi_3 \Pi_4$ is greater than or equal to the size of Q_3 . In other words,

$$1 + 1 = r_3 + r_4 \geq q_3 = 3. \quad (7)$$

Contradiction.

We remark that

$$r_3 + r_4 + 3 + 3 = t_1 + t_2 + t_3 \text{ and } q_3 + 3 + 3 = p_1 + p_2 + p_3.$$

Thus Eq. (7) actually contradicts Eq. (6).

Proof of Theorem 3.1 part (1). Now we turn to the general case. There is an index l such that $p_1 + \cdots + p_l > t_1 + \cdots + t_l$. Suppose there exists $w \in W(P_{\mu^\top}) \setminus W(G)$ such that $w(\Delta_\lambda) \subset \Phi^-$. We shall derive a contradiction. If $l = m$, then $r_1 = b > m$. The number of cycles in $w \in W(P) \setminus W(G)$ is at most $r + 1 - r_1$, but the number of simple roots in Δ_λ is $r - m$. This contradicts Lemma 3.6 (compare with Example 3.12).

If $l < m$, then

$$R_2 \sqcup \cdots \sqcup R_{t_{l+1}} \subset Q_2 \sqcup \cdots \sqcup Q_{t_{l+1}}. \quad (8)$$

Otherwise, $R_2 \sqcup \cdots \sqcup R_{t_{l+1}}$ contains a simple root α_k in $Q_{t_{l+1}+1} \sqcup \cdots \sqcup Q_{t_n}$. Without loss of generality, we assume that $\alpha_k \in Q_{t_{l+1}+1}$. This means that

$$\alpha_{k+1} \in Q_{t_{l+1}}, \dots, \alpha_{k+t_{l+1}-1} \in Q_2.$$

On the other hand, by Lemma 3.10, we deduce that

$$\alpha_{k+1} \in R_2 \sqcup \cdots \sqcup R_{t_{l+1}-1}, \dots, \alpha_{k+t_{l+1}-2} \in R_2.$$

This forces that $\alpha_{k+t_{l+1}-1} \notin \Delta_\lambda$, which is impossible from our assumption. Thus Eq. (8) holds and

$$Q_{t_{l+1}+1} \sqcup \cdots \sqcup Q_n \subset R_{t_{l+1}+1} \sqcup \cdots \sqcup R_a.$$

By Lemma 3.6,

$$q_{t_{l+1}+1} + \cdots + q_n \leq r_{t_{l+1}+1} + \cdots + r_a.$$

However, by Lemma 3.2,

$$\begin{aligned} & l \cdot t_{l+1} + q_{t_{l+1}+1} + \cdots + q_n \\ & \geq p_1 + \cdots + p_l \\ & > t_1 + \cdots + t_l \\ & = l \cdot t_{l+1} + r_{t_{l+1}+1} + \cdots + r_a. \end{aligned}$$

This is a contradiction (compare with Example 3.13). \square

3.4. Proof of Theorem 3.1 part (2). We need to show that there exists a unique $w_\mu = \Pi_2 \cdots \Pi_a \in W(P_{\mu^\top}) \setminus W(G)$ such that $w_\mu(\Delta_\mu)$ is contained in $\Phi^- - \Phi_{\mu^\top}^-$; and $w(\Delta_\mu) \cap \Phi^+ \neq \emptyset$ for all $w \neq w_\mu$. Before giving the proof we give an example to explain the idea.

Example 3.14. Let $\mu = \lambda = (433)$, $\mu^\top = (3331)$. The associated Young tableaux are

$$\mu : \begin{array}{c} r_1 \quad r_2 \quad r_3 \quad r_4 \\ t_1 \quad \boxed{} \quad \boxed{k_3} \quad \boxed{k_6} \quad \boxed{k_9} \\ t_2 \quad \boxed{} \quad \boxed{k_4} \quad \boxed{k_7} \\ t_3 \quad \boxed{} \quad \boxed{k_5} \quad \boxed{k_8} \end{array} \quad \lambda : \begin{array}{c} q_1 \quad q_2 \quad q_3 \quad q_4 \\ p_1 \quad \boxed{} \quad \boxed{\alpha_3} \quad \boxed{\alpha_2} \quad \boxed{\alpha_1} \\ p_2 \quad \boxed{} \quad \boxed{\alpha_6} \quad \boxed{\alpha_5} \\ p_3 \quad \boxed{} \quad \boxed{\alpha_9} \quad \boxed{\alpha_8} \end{array}$$

We have

$$R_2 = Q_2 = \{\alpha_3, \alpha_6, \alpha_9\}, \quad R_3 = Q_3 = \{\alpha_2, \alpha_5, \alpha_8\}, \quad R_4 = Q_4 = \{\alpha_1\}.$$

This allows us to find the desired w_μ inductively. Notice that a negative root is in $\Phi^- - \Phi_{\mu^\top}^-$ if and only if the coefficients of α_3, α_6 or α_9 is -1 .

By Lemma 3.3, the only choice for Π_4 corresponds to $k_9 = 1$. It is easy to check that $s_{987654321}(\alpha_1) = -(\alpha_1 + \cdots + \alpha_9)$, and therefore $w(\alpha_1) \in \Phi^- - \Phi_{\mu^\top}^-$ since the coefficient of α_9 is -1 .

We then need to find $\Pi_2\Pi_3$ such that $\Pi_2\Pi_3(w) < 0$ for all $w \in \Pi_4(Q_2 \cup Q_3)$. Notice that this is equivalent to the same problem, but with a smaller rank. To be more precise, we may naturally view $\Pi_2\Pi_3$ as an element in $W(P_{\mu'^\top}) \setminus W(\text{GL}_9)$, where $\mu' = (333)^\top = (333)$. (In terms of the Young diagram, we only need to delete the last column.) On the other hand, $\Pi_4(Q_2 \cup Q_3) = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_7, \alpha_8\}$, which are exactly the simple roots we need to consider in this smaller rank case.

$$\mu' : \begin{array}{c} r_1 \quad r_2 \quad r_3 \\ \begin{array}{|c|c|c|} \hline t_1 & & k_3 \quad k_6 \\ \hline t_2 & & k_4 \quad k_7 \\ \hline t_3 & & k_5 \quad k_8 \\ \hline \end{array} \end{array} \quad \lambda' : \begin{array}{c} q_1 \quad q_2 \quad q_3 \\ \begin{array}{|c|c|c|} \hline p_1 & & \alpha_2 \quad \alpha_1 \\ \hline p_2 & & \alpha_5 \quad \alpha_4 \\ \hline p_3 & & \alpha_8 \quad \alpha_7 \\ \hline \end{array} \end{array}$$

Now we claim that there is only one choice for Π_3 , corresponding to $(k_6, k_7, k_8) = (1, 4, 7)$. Recall that we need to find Π_3 which sends α_1, α_4 , and α_7 to negative roots. Indeed, if $k_8 < 7$, then $\pi_{k_8}(\alpha_7) = \alpha_6$ and by Lemma 3.7, $\Pi_3(\alpha_7) = \alpha_4$, which is positive; if $k_8 = 8$, then $w(\alpha_7) > 0$; if $k_8 > 8$, then $\pi_{k_8}(\alpha_1, \alpha_4, \alpha_7) = \alpha_1, \alpha_4, \alpha_7$, and $\pi_{k_6}\pi_{k_7}$ cannot send $\alpha_1, \alpha_4, \alpha_7$ to negative roots. Otherwise, this contradicts Lemma 3.6 and $Q_3 = R_3$. Similarly we deduce that $k_7 = 4$ and $k_6 = 1$. One may also notice that the coefficient of α_6 in $\Pi_3(\alpha_1), \Pi_3(\alpha_4)$, and $\Pi_3(\alpha_7)$ are all -1 , thus the coefficients of α_6 in $w(\alpha_1), w(\alpha_4)$, and $w(\alpha_7)$ are also -1 . This shows that they are in $\Phi^- - \Phi_{\mu'^\top}^-$.

Lastly, we need to choose Π_2 such that $\Pi_2(w) < 0$ for all $w \in \Pi_3\Pi_4(Q_2)$. We may similarly see that this is equivalent to the problem with $\mu''^\top = (33)$. The corresponding Young tableaux are obtained by deleting the last row from μ' and we see that $(k_3, k_4, k_5) = (1, 3, 5)$. In other words, the unique choice for w_μ is

$$(s_{321}s_{43}s_5)(s_{654321}s_{7654}s_{87})(s_{987654321}).$$

Proof of Theorem 3.1 part (2). Suppose $w(\alpha) < 0$ for all $\alpha \in \Delta_\mu$. We prove the result by induction on a . Clearly $R_i = Q_i$ for $2 \leq i \leq a$.

We first show that there is a unique choice for Π_a . Let $x = r_1 + \cdots + r_{a-1} - 1$. Then $(k_{x+1}, \dots, k_{x+r_a})$ are the entries in the last column of μ . Let $(\alpha_{l_1}, \dots, \alpha_{l_{r_a}})$ be the simple roots in the last column of λ . Indeed, one can check that $l_i = 1 + (i-1)p_1$. We claim that

$$(k_{x+1}, \dots, k_{x+r_a}) = (l_1, \dots, l_{r_a}).$$

This is clear if $r_a = 1$. Now we argue by induction on r_a . If $k_{x+r_a} < l_{r_a}$, then by Lemma 3.7 $\Pi_a(\alpha_{l_{r_a}}) = \alpha_{l_{r_a}-r_a}$, which is positive; if $k_{x+r_a} = l_{r_a} + 1$, then $w(\alpha_{r_a}) > 0$; if $k_{x+r_a} > l_{r_a} + 1$, then $(\alpha_{l_1}, \dots, \alpha_{l_{r_a}})$ are invariant under $\pi_{k_{x+r_a}}$, and $\pi_{k_{x+1}} \cdots \pi_{k_{x+r_a-1}}$ cannot send $(\alpha_{l_1}, \dots, \alpha_{l_{r_a}})$ to negative roots. Thus, $k_{x+r_a} = l_{r_a}$. Notice that for this choice, $(\alpha_{l_1}, \dots, \alpha_{l_{r_a-1}})$ are invariant under $\pi_{k_{x+r_a}}$. This allows us to define Π_a inductively.

The above argument also shows that Π_a is unique. Moreover, it is easy to check that the coefficients of α_{x+1} in $\Pi_a(\alpha_{l_i})$'s are -1 , thus the coefficients in $w(\alpha_{l_i})$'s are also -1 . This implies that $w(\alpha_{l_i}) \in \Phi^- - \Phi_{\mu'^\top}^-$.

Now notice that, $\Pi_2 \cdots \Pi_{a-1}$ may be viewed as an element in $W(P_{\mu'^\top}) \setminus W(\text{GL}_{r+1-r_a})$, where $\mu' = (r_1 \cdots r_{a-1})^\top$. The operation on the Young diagram is simply deleting the last column. On the other hand, under the action of Π_a , the simple roots in the last column are sent to negative roots. For $1 \leq i \leq r_a$, the indices in the i th row drop by i ; and the indices

in the remaining rows drop by r_a . Thus we reduce the problem to GL_{r+1-r_a} with partition $(r_1 \cdots r_{a-1})^\top$, which is of smaller rank. This allows us to define $w = \Pi_2 \cdots \Pi_a$ inductively and uniquely. \square

4. DEGENERATE EISENSTEIN SERIES

Let F be a number field. Let \mathbb{A} be its adele ring. Let $\mu = (t_1 \cdots t_b)$ be a partition of $r+1$ with $t_1 \geq \cdots \geq t_b > 0$. Let $\mu^\top = (r_1 \cdots r_a)$ denote the transpose of μ . Let $P = P_{\mu^\top}$. Let δ_P be the modular quasicharacter of G with respect to P . We denote by $E_\mu(g, \underline{s})$ the Eisenstein series which corresponds to the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \delta_P^{\underline{s}}$ (this depends on a choice of test vectors, but we suppress this from the notation). Here $\underline{s} = (s_1, \dots, s_a)$ denotes a multi-complex variable.

We now define semi-Whittaker coefficients. Let $\lambda = (p_1 \cdots p_m)$ be a general partition of $r+1$. Fix a nontrivial additive character $\psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$. Let $\psi_\lambda : U(F) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$ be the character such that it acts as ψ on the root subgroups associated with $\alpha \in \Delta_\lambda$, and acts trivially otherwise. Given an automorphic form f on $\mathrm{GL}_{r+1}(\mathbb{A})$, we define a λ -semi-Whittaker coefficient of f as the integral

$$\int_{U(F) \backslash U(\mathbb{A})} f(ug) \psi_\lambda(u) \, du.$$

Theorem 4.1. *Suppose $\mathrm{Re}(s_i) \gg 0$.*

(1) *If there is an index l such that $p_1 + \cdots + p_l > t_1 + \cdots + t_l$, then*

$$\int_{U(F) \backslash U(\mathbb{A})} E_\mu(ug, \underline{s}) \psi_\lambda(u) \, du = 0$$

for all choices of data.

(2) *The semi-Whittaker coefficient*

$$\int_{U(F) \backslash U(\mathbb{A})} E_\mu(ug, \underline{s}) \psi_\mu(u) \, du$$

is nonzero for some choice of data.

Proof. For $\mathrm{Re}(s_i) \gg 0$ we unfold the Eisenstein series. Thus we need to study the space $P \backslash G/U$, and analyze the contribution from each representative. By the Bruhat decomposition, we identify $P \backslash G/U$ with $W(P) \backslash W(G)$. To prove part (1), it suffices to show that for every $w \in W(P) \backslash W(G)$, there is a $u \in U$ such that $\psi_\lambda(u) \neq 1$ and $wuw^{-1} \in U$. This follows from Theorem 3.1 part (1).

To prove part (2), again by Theorem 3.1 part (2), we see that the only contribution comes from w_μ . Thus,

$$\int_{U(F) \backslash U(\mathbb{A})} E_\mu(ug, \underline{s}) \psi_\lambda(u) \, du = \int_{U_{w_\mu}(\mathbb{A})} f(w_\mu ug, \underline{s}) \psi_\lambda(u) \, du,$$

where U_{w_μ} is the subgroup of U which corresponds to the roots $\alpha > 0$ such that $w_\mu(\alpha) < 0$. The right-hand side is factorizable, and its value is a ratio of zeta functions. For $\mathrm{Re}(s_i) \gg 0$ it is nonzero. This completes the proof. \square

The corresponding local result also holds and can be proved similarly; see also [18].

Theorem 4.2. *Let F be a non-Archimedean local field.*

(1) *If there is an index l such that $p_1 + \cdots + p_l > t_1 + \cdots + t_l$, then*

$$\dim \operatorname{Hom}_{U(F)}(\operatorname{Ind}_{P(F)}^{G(F)} \delta_P^s, \psi_\lambda) = 0.$$

(2) *If $\lambda = \mu$, then*

$$\dim \operatorname{Hom}_{U(F)}(\operatorname{Ind}_{P(F)}^{G(F)} \delta_P^s, \psi_\mu) = 1.$$

5. UNIPOTENT ORBITS AND FOURIER COEFFICIENTS

5.1. Fourier coefficients associated with unipotent orbits. Given a unipotent orbit, we can associate a set of Fourier coefficients. General references for unipotent orbits are Carter [2] and Collingwood and McGovern [5]. For the local version of this association see [17, 18]. For global details see Jiang and Liu [13] and Ginzburg [6, 8].

We work with the global setup. Let F be a number field, and \mathbb{A} be its adele ring. Fix a nontrivial additive character $\psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$. The unipotent orbits of GL_r are parameterized by partitions of r . Let $\mathcal{O} = (p_1 \cdots p_k)$ with $p_1 + \cdots + p_k = r$ be a unipotent orbit. We shall always assume $p_1 \geq p_2 \geq \cdots \geq p_k > 0$. To each p_i we associate the diagonal matrix

$$\operatorname{diag}(t^{p_i-1}, t^{p_i-3}, \dots, t^{3-p_i}, t^{1-p_i}).$$

Combining all such diagonal matrices and arranging them in decreasing order of the powers of t , we obtain a one-dimensional torus $h_{\mathcal{O}}(t)$. For example, if $\mathcal{O} = (3^2 1)$, then

$$h_{\mathcal{O}}(t) = \operatorname{diag}(t^2, t^2, 1, 1, 1, t^{-2}, t^{-2}).$$

The one-dimensional torus $h_{\mathcal{O}}(t)$ acts on U by conjugation. Let α be a positive root and $x_\alpha(a)$ be the one-dimensional unipotent subgroup in U corresponding to the root α . There is a nonnegative integer m such that

$$h_{\mathcal{O}}(t)x_\alpha(a)h_{\mathcal{O}}(t)^{-1} = x_\alpha(t^m a). \quad (9)$$

On the subgroups $x_\alpha(a)$ which correspond to negative roots α , the torus $h_{\mathcal{O}}(t)$ acts with non-positive powers.

Given a nonnegative integer l , we denote by $U_l(\mathcal{O})$ the subgroup of U generated by all $x_\alpha(a)$ satisfying the Eq. (9) with $m \geq l$. We are mainly interested in $U_l(\mathcal{O})$ where $l = 1$ or $l = 2$.

Let

$$M(\mathcal{O}) = T \cdot \langle x_{\pm\alpha}(a) : h_{\mathcal{O}}(t)x_\alpha(a)h_{\mathcal{O}}(t)^{-1} = x_\alpha(a) \rangle.$$

The algebraic group $M(\mathcal{O})$ acts by conjugation on the abelian group $U_2(\mathcal{O})/U_3(\mathcal{O})$. If the ground field is algebraically closed, then under this action of $M(\mathcal{O})$ on the group $U_2(\mathcal{O})/U_3(\mathcal{O})$, there is an open orbit. Denote a representative of this orbit by u_2 . It follows from the general theory that the connected component of the stabilizer of this orbit inside $M(\mathcal{O})$ is a reductive group. Denote by $\operatorname{Stab}_{\mathcal{O}}^0$ this connected component of the stabilizer of u_2 .

The group $M(\mathcal{O})(F)$ acts on the group of all characters of $U_2(\mathcal{O})(F) \backslash U_2(\mathcal{O})(\mathbb{A})$. Consider the subset of all characters such that over the algebraic closure, the connected component of the stabilizer inside $M(\mathcal{O})(F)$ is equal to $\operatorname{Stab}_{\mathcal{O}}^0$. We denote such a character by $\psi_{U_2(\mathcal{O})}$.

Given an automorphic function $\varphi(g)$ on $\mathrm{GL}_r(\mathbb{A})$, the Fourier coefficients we want to consider are

$$\int_{U_2(\mathcal{O})(F) \backslash U_2(\mathcal{O})(\mathbb{A})} \varphi(ug) \psi_{U_2(\mathcal{O})}(u) du.$$

In this way, we associate with each unipotent orbit \mathcal{O} a set of Fourier coefficients. When the partition is $\mathcal{O} = (r)$, the Fourier coefficients associated to \mathcal{O} are the Whittaker coefficients.

Let us recall the partial ordering defined on the set of unipotent orbits. Given $\mathcal{O}_1 = (p_1 \cdots p_k)$ and $\mathcal{O}_2 = (q_1 \cdots q_l)$, we say that $\mathcal{O}_1 \geq \mathcal{O}_2$ if $p_1 + \cdots + p_i \geq q_1 + \cdots + q_i$ for all $1 \leq i \leq l$. If \mathcal{O}_1 is not greater than \mathcal{O}_2 and \mathcal{O}_2 is not greater than \mathcal{O}_1 , we say that \mathcal{O}_1 and \mathcal{O}_2 are not comparable.

Definition 5.1. Let π be an automorphic representation of $\mathrm{GL}_r(\mathbb{A})$. Let $\mathcal{O}(\pi)$ denote the set of unipotent orbits of GL_r defined as follows. A unipotent orbit \mathcal{O} is in $\mathcal{O}(\pi)$ if π has a nonzero Fourier coefficient which is associated with the unipotent orbit \mathcal{O} , and for all $\mathcal{O}' > \mathcal{O}$, π has no nonzero Fourier coefficient associated with \mathcal{O}' .

Definition 5.1 is described in the global setup. One may have a similar definition in the local context where Fourier coefficients are replaced by twisted Jacquet modules. We omit the details.

Remark 5.2. It is expected that for any automorphic representation π , the set $\mathcal{O}_G(\pi)$ is a singleton (see [6] Conjecture 5.4). In this paper, the notation $\mathcal{O}_G(\pi) = \mu$ means that the set $\mathcal{O}_G(\pi)$ is a singleton, consisting of the orbit μ only.

5.2. Connection to Semi-Whittaker Coefficients. There is a strong relation between semi-Whittaker coefficients and Fourier coefficients associated with unipotent orbits. We prove a global version and state a local version in this section. *Notice: the results and notations are independent of the rest of this paper.*

5.2.1. Statements. Let $\mathcal{O} = (p_1 \cdots p_m)$ be a unipotent orbit for GL_r . Let $\lambda = (q_1 \cdots q_n)$ be a general partition of r . Define a character $\psi_\lambda : U(F) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$ as in the previous sections.

Proposition 5.3. *Let π be an automorphic representation on $\mathrm{GL}_r(\mathbb{A})$. The following are equivalent:*

- (1) *The unipotent orbit attached to π is $\mathcal{O}(\pi) = \mathcal{O}$.*
- (2) *If $q_1 + \cdots + q_i \geq p_1 + \cdots + p_i$ for some i , then*

$$\int_{U(F) \backslash U(\mathbb{A})} f(ug) \psi_\lambda(u) du = 0$$

for all choices of data; and if $\lambda = \mathcal{O}$, then

$$\int_{U(F) \backslash U(\mathbb{A})} f(ug) \psi_\lambda(u) du$$

is nonzero for some choice of data.

The corresponding local version is also true.

Proposition 5.4. *Let F be a non-Archimedean local field. Let π be a smooth representation on $\mathrm{GL}_r(F)$. The following are equivalent:*

- (1) The unipotent orbit attached to π is $\mathcal{O}(\pi) = \mathcal{O}$.
- (2) If $q_1 + \dots + q_i \geq p_1 + \dots + p_i$ for some i , then the twisted Jacquet module $J_{U(F), \psi_\lambda}(\pi) = 0$; and if $\lambda = \mathcal{O}$, then $J_{U(F), \psi_\lambda}(\pi) \neq 0$.

Moreover, if p_i 's have the same parity, then

$$J_{U(F), \psi_{\mathcal{O}}}(\pi) \cong J_{U(F), \psi_{U_2(\mathcal{O})}}(\pi).$$

Remark 5.5. When $\mathcal{O} = (n^a b)$, these results are proved in [1] Theorem 7.7 and 7.8.

Remark 5.6. A local result of similar flavor can also be found in Gomez, Gourevitch and Sahi [11] Theorem E.

We present a proof of the global result in this section. The local case is similar and we omit the details. To save space, we write $[V] = V(F) \backslash V(\mathbb{A})$ for a subgroup V of U .

5.2.2. *Part (2) implies part (1).* We first show that part (2) implies part (1). To prove the vanishing part, we need to show that any unipotent orbit which is greater than or not comparable with \mathcal{O} does not support any Fourier coefficient. We establish several lemmas.

Let $\mathcal{O}' = (q_1 \dots q_n)$ be a unipotent orbit which is greater than or not comparable with \mathcal{O} . Then there exists i such that $q_1 + \dots + q_i > p_1 + \dots + p_i$. Let $m' \geq q_1 + \dots + q_i$. Let $\underline{\epsilon} = (\epsilon_{q_1 + \dots + q_i}, \dots, \epsilon_{m'})$ with $\epsilon_{q_1 + \dots + q_i}, \dots, \epsilon_{m'} \in F$. Let $V_{m'}$ be the unipotent radical with Levi $\mathrm{GL}_1^{m'} \times \mathrm{GL}_{r-m'}$. The semi-Whittaker character $\psi_{(q_1 \dots q_i)}$ may be also viewed as a character on $[V_{m'}]$. Define characters $\psi_{(q_1 \dots q_i), \underline{\epsilon}} : [V_{m'}] \rightarrow \mathbb{C}^\times$,

$$\psi_{(q_1 \dots q_i), \underline{\epsilon}}(v) = \psi_{(q_1 \dots q_i)}(v) \cdot \psi \left(\sum_{j=q_1 + \dots + q_i}^{m'} \epsilon_j v_{j,j+1} \right).$$

Lemma 5.7. *The integral*

$$\int_{[V_{m'}]} f(ug) \psi_{(q_1 \dots q_i), \underline{\epsilon}}(u) du = 0$$

for all choices of data. In particular,

$$\int_{[V_{q_1 + \dots + q_i}]} f(ug) \psi_{(q_1 \dots q_i)}(u) du = 0$$

for all choices of data.

Proof. We prove this by induction on $r - m'$. If $r - m' = 0$, then the integral involved is just a semi-Whittaker coefficient. The result follows from part (2). Assume the result is true for $r - m'$. Then we prove it for $r - m' - 1$ if $m' - 1 \geq q_1 + \dots + q_i$. Let $R_{m'}$ be the unipotent subgroup such that $u_{j,l} = 0$ unless $j = m'$. Expand the integral along this unipotent subgroup. Both nontrivial and trivial orbits give zero by induction. This proves the result. □

Define $U'_{(q_1 \dots q_i), k}$ as the subgroup of $V_{q_1 + \dots + q_i}$ such that $u \in U'_{(q_1 \dots q_i), k}$ if $u_{j,l} = 0$ for $j = q_1 + \dots + q_k, q_1 + \dots + q_{k+1}, \dots, q_1 + \dots + q_i$. We also define $U'_{(q_1 \dots q_i), i+1} = V_{q_1 + \dots + q_i}$. The character $\psi_{(q_1 \dots q_i)}$ is still a character on $U'_{(q_1 \dots q_i), k}$.

Lemma 5.8. *For $1 \leq k \leq i$, the integral*

$$\int_{[U'_{(q_1 \dots q_i), k}]} f(ug) \psi_{(q_1 \dots q_i)}(u) \, du = 0 \quad (10)$$

for all choices of data.

Proof. We prove this by induction on $i - k$. When $i - k = 0$, the group $U'_{(q_1 \dots q_i), k}$ is actually $V_{q_1 + \dots + q_i - 1}$. Expand the integral (10) along $R_{q_1 + \dots + q_i}$. Both trivial and nontrivial orbits give zero by Lemma 5.7. This establishes the case $i = k$. Assume the result is true for $i - (k + 1)$. Then we prove it for k . Indeed, we may write

$$\begin{aligned} & \int_{[U'_{(q_1 \dots q_i), k}]} f(ug) \psi_{(q_1 \dots q_i)}(u) \, du \\ &= \int_{[U'_{(q_{k+1} \dots q_i), i-k-1}]} \int_{[V_{q_1 + \dots + q_k - 1}]} f(uvg) \psi_{(q_1 \dots q_k)}(u) \, du \, \psi_{(q_{k+1} \dots q_i)}(v) \, dv. \end{aligned}$$

Here, we view $U'_{(q_{k+1} \dots q_i), i-k-1}$ as a subgroup of U via $g \mapsto \text{diag}(I_{q_1 + \dots + q_k}, g)$. Expand the inner integral along $R_{q_1 + \dots + q_k}$. The trivial orbit is zero by induction; the nontrivial orbit gives the Fourier coefficient for a larger partition $(q_1, \dots, q_{k-1}, q_k + q_{k+1}, \dots, q_i)$, which is also zero by induction. Thus the integral (10) is zero. \square

Now we recall a corollary of root exchange. The local version is proved in [1] Lemma 6.5. The global version can be proved analogously.

Lemma 5.9. *Let π be an automorphic representation on $\text{GL}_r(\mathbb{A})$. Let $\mathcal{O} = (p_1 1^{n-p_1})$. Then*

$$\int_{[U_2(\mathcal{O})]} f(ug) \psi_{U_2(\mathcal{O})}(u) \, du = 0$$

for all choices of data if and only if

$$\int_{[V_{p_1-1}]} f(ug) \psi_{(p_1)}(u) \, du = 0$$

for all choices of data.

By applying this lemma repeatedly, we obtain the following result.

Lemma 5.10. *Let $\mathcal{O} = (q_1 \dots q_i 1^{r-q_1-\dots-q_i})$. Then*

$$\int_{[U_2(\mathcal{O})]} f(ug) \psi_{U_2(\mathcal{O})}(u) \, du = 0$$

for all choices of data if and only if

$$\int_{[U'_{(q_1 \dots q_i), 1}]} f(ug) \psi_{(q_1 \dots q_i)}(u) \, du = 0.$$

for all choices of data.

Now we are ready to prove the general case. Let $\mathcal{O}'' = (q_1 \cdots q_i 1^{r-q_1-\cdots-q_i})$. We then have

$$\int_{[U_2(\mathcal{O}')] } f(ug) \psi_{U_2(\mathcal{O}')} (u) du = \int_X \int_{[U_2(\mathcal{O}'')] } f(uxg) \psi_{(q_1 \cdots q_i)} (u) \psi_X(x) dx \quad (11)$$

where X is some subgroup which we don't need to specify. From Lemmas 5.8 and 5.10, we see that the inner integral in Eq. (11) is zero. This proves the vanishing part.

To prove the nonvanishing part, it suffices to show that

$$\int_{[U'_{(p_1 \cdots p_m), 1}]} f(ug) \psi_{(p_1 \cdots p_m)} (u) du \neq 0 \quad (12)$$

for some choice of data. Notice that the integral in Eq. (12) is

$$\int_{[U'_{(p_2 \cdots p_k), 1}]} \int_{[U_{(p_1)}]} f(vug) \psi_{(p_1)} (v) \psi_{U_2(p_2 \cdots p_k)} (u) dv du$$

where $U'_{(p_2 \cdots p_k), 1}$ is viewed as a subgroup of U via $g \mapsto \text{diag}(I_{p_1}, g)$. Expand the inner integral along the subgroup R_{p_1} . The nontrivial orbit gives 0 since that corresponds to the orbit $(p_1 + p_2, p_3 \cdots p_k)$. Thus the above integral equals

$$\int_{[U'_{(p_1 \cdots p_m), 2}]} f(ug) \psi_{(p_1 \cdots p_m)} (u) du.$$

We now repeat this process by expanding along $R_{p_1+p_2}, \dots, R_{p_1+\cdots+p_{m-1}}$. This shows that the above integral is

$$\int_{[U]} f(ug) \psi_{(p_1 \cdots p_m)} (u) du. \quad (13)$$

This is nonzero for some choice of data from the assumption on semi-Whittaker coefficients.

5.2.3. Part (1) implies part (2). We first consider the vanishing part. Let $\lambda = (q_1 \cdots q_n)$ such that $q_1 + \cdots + q_i > p_1 + \cdots + p_i$ for some i . We rearrange $(q_1 \cdots q_i)$ into non-increasing order to obtain $(q'_1 \cdots q'_i)$. Consider the unipotent orbit $\mathcal{O}' = (q'_1 \cdots q'_i 1 \cdots 1)$. This unipotent orbit does not support any Fourier coefficient. By conjugating the integral with a suitable Weyl group element and applying Lemma 5.9, we see that

$$\int_{[U'_{(q_1 \cdots q_i), 1}]} f(ug) \psi_{(q_1 \cdots q_i)} (u) du = 0 \quad (14)$$

for all choices of data.

Lemma 5.11. *For $1 \leq k \leq i + 1$, the integral*

$$\int_{[U'_{(q_1 \cdots q_i), k}]} f(ug) \psi_{(q_1 \cdots q_i)} (u) du = 0$$

for all choices of data. In particular,

$$\int_{[V_{q_1+\dots+q_i}]} f(ug)\psi_{(q_1\dots q_i)}(u) du = 0 \quad (15)$$

for all choices of data.

Proof. We prove this by induction on k . The case $k = 1$ follows from Eq. (14). Suppose the result is true for $k - 1$, and we prove it for k . Expand the integral

$$\int_{[U'_{(q_1\dots q_i),k-1}]} f(ug)\psi_{(q_1\dots q_i)}(u) du$$

along the unipotent subgroup $R_{q_1+\dots+q_{k-1}}$. The nontrivial orbit gives zero, since the coefficients correspond to a larger orbit and we can apply induction. Thus we only have the trivial orbit and

$$\int_{[U'_{(q_1\dots q_i),k}]} f(ug)\psi_{(q_1\dots q_i)}(u) du = \int_{[U'_{(q_1\dots q_i),k-1}]} f(ug)\psi_{(q_1\dots q_i)}(u) du = 0$$

for all choices of data. □

To prove the vanishing statement in part (2), notice that

$$\int_{[U]} f(ug)\psi_{\lambda}(u) du \quad (16)$$

contains Eq. (15) as an inner integral. Thus Eq. (16) is zero for all choices of data.

Finally, we prove the nonvanishing statement in part (2). This follows from the same argument as in the previous section. Once we have the vanishing result, Eq. (13) is nonzero for some choice of data if and only if Eq. (12) is nonzero for some choice of data. This finishes the proof.

5.3. Main result. The main result in this paper is the following theorem.

Theorem 5.12. *With the notations in Section 4, suppose that $\text{Re}(s_i) \gg 0$. We then have*

$$\mathcal{O}_G(E_{\mu}(g, \underline{s})) = \mu.$$

The theorem follows from Theorem 4.1 and Proposition 5.3. This confirms Conjecture 5.1 in Ginzburg [6]. We also remark that the nonvanishing part is also proved in Ginzburg [7] Section 3, Proposition 1. This also implies that any unipotent orbit of general linear groups can occur as the unipotent orbit attached to a specific automorphic representation.

Notice that by Theorem 4.2 and Proposition 5.4, a local analogue to Theorem 5.12 is also true.

Theorem 5.13. *Let F be a non-Archimedean local field. Then*

$$\mathcal{O}_G(\text{Ind}_{P(F)}^{G(F)} \delta_P^s) = \mu.$$

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